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LINEAR THEORY OF THE PROPAGATION OF INTERNAL WAVE BEAMS IN AN ARBITRARILY STRATIFIED LIQUID

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Beams of harmonic internal waves in a liquid with smoothly changing stratification are calculated in the Boussinesq approximation taking into account the effects of diffusion and viscosity. A procedure of local reduction of the beam in a medium with an arbitrary smooth stratification to the case of an exponentially stratified liquid is constructed. The coefficient of energy losses in the case of beam reflection on the critical level is calculated. Parameters of internal boundary flows with split scales of velocity and density that are formed by a wave beam on discontinuities of the buoyancy frequency and its higher derivatives are determined.

Introduction. Two types of internal waves are traditionally distinguished: waves localized on density-discontinuity layers and volumetric waves, which propagate over the entire thickness of the liquid [1]. Their properties are studied in the approximation of perfect [2] or viscous [3] liquids. In a medium with an exponential distribution of density, the waves propagate along radius-vectors whose slope to the horizon θ is determined as the ratio of the wave frequency ω to the buoyancy frequency N: $\sin\theta = \omega/N$. In a medium with an arbitrary stratification, regular waves exist in regions where $\omega < N$. As the critical level $\omega = N$ is approached, the wave beam deflects from the vertical, the wave vectors become horizontal, the further propagation of the waves is impossible, and beam reflection occurs [4].

Allowance for viscosity and diffusion substantially changes the description of internal waves. In this case, a compact source generates a field of internal waves that is regular over the entire space [3]. Significant disturbances are concentrated in narrow wave beams that contain one and a half to two spatial oscillations. Asymptotic solutions are in agreement with measurements and observations of internal waves under laboratory conditions even near the source [5].

When internal-wave beams are reflected from a flat rigid surface, boundary flows with split scales of velocity and density arise owing to viscosity and diffusion effects [6–8]. A marked portion of energy of the incident-wave beam is converted to a boundary flow periodic in time [9].

In most cases, models of internal waves are constructed for smooth distributions of density [10]. Under natural conditions, a fine structure of the medium with expressed discontinuity layers of density and its derivatives (up to high-order derivatives) is observed. In this connection, it is of interest to study the effect of discontinuity in the gradient of density and its higher derivatives (in the absence of a jump in density) on the propagation of internal-wave beams taking into account the effects of viscosity and diffusion and to calculate the corresponding disturbances originating on inhomogeneities of stratification. By analogy with [8, 9], the wave beam can be expected to transform into different forms of spatially localized motions with nonwave natural scales.

The purpose of the present paper is a dynamic consideration of the problem of the propagation of internal-wave beams in an arbitrarily stratified medium taking into account the dissipative effects (viscosity and diffusion), wave reflection on the critical level, a calculation of energy losses due to this reflection, and a study of propagation of the beams in the vicinity of density derivative discontinuities in the medium.

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1. Governing Equations. The following mathematical formulation of the problem of the propagation of wave beams in a medium with an arbitrary stratification is used. Let an incompressible viscous liquid with an arbitrary stratification of reduced salinity $s_0(z)$, which includes the salt compression coefficient of the diffusing salt component be located in a gravity-force field with acceleration g opposed to the vertical axis z. The dependence of the undisturbed density on the vertical coordinate z is $\rho_0(z) = \rho_{00}[1 + s_0(z)]$. We study a two-dimensional problem, in which, besides z all quantities depend only on the horizontal coordinate x. We consider monochromatic waves with a time dependence of the form $e^{-i\omega t}$, which is omitted in what follows.

In this case, in the Boussinesq approximation we can write the linearized system of equations [13] that describes the motion of the liquid:

$$-i\omega v_x = -\frac{1}{\rho_{00}}\frac{\partial P}{\partial x} + \nu\Delta v_x, \qquad -i\omega v_z = -\frac{1}{\rho_{00}}\frac{\partial P}{\partial z} + \nu\Delta v_z - sg,$$

$$-i\omega s + v_z\frac{d s_0}{d z} = D\Delta s, \qquad \frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0.$$

Here v_x and v_z are components of the velocity of liquid particles, P and s are the variable pressure and salinity, ν and D are the kinematic viscosity and the salt diffusivity, respectively, and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial z^2$ is a two-dimensional Laplacian.

This system leads to the following equation for the vertical displacements of the particles h, which are related to the vertical velocity as $v_z = -i\omega h$,

$$\Big[(\omega - iD\Delta)(\omega - i\nu\Delta)\Delta - N^2(z)\frac{\partial^2}{\partial x^2}\Big]h = 0, \tag{1.1}$$

where $N^2(z) = -(g/\rho_0(z))(d\rho_0(z)/dz)$ is the square of the buoyancy frequency, which generally depends on z.

Beams of internal waves from a localized source propagate in four directions [3]: right and up, right and down, left and up, and left and down. Without limiting the generality, we consider only beams that propagate to the right, for which the horizontal component of the wave vector is positive. This allows us to seek a solution of Eq. (1.1) in the form

$$h = \int_0^\infty f(z,k)e^{ikx}\,dk. \tag{1.2}$$

Substituting this expression into (1.1), we obtain the following ordinary differential equation for $f(z,k)$ (differentiation is performed only with respect to z):

$$\nu D f^{(6)} + [i\omega(\nu + D) - 3\nu Dk^2]f^{(4)} - [\omega^2 + 2i\omega(\nu + D)k^2$$
$$- 3\nu Dk^4]f^{(2)} + [\omega^2 - N^2(z) + i\omega(\nu + D)k^2 - \nu Dk^4]f = 0, \tag{1.3}$$

The properties of this equation for the case of an exponentially stratified medium are studied in [7–9]. The Arabic numerals in brackets denote the order of derivatives.

Equation (1.3) is a singularly disturbed sixth-order differential equation (a small coefficient at the highest derivative). For $\nu = D = 0$, it transforms into a traditional second-order equation that describes internal waves propagating in a medium without dissipation [2, 3]. The presence of nonzero viscosity and diffusion lead to decay of waves because of energy dissipation and entrainment of undisturbed liquid into wave motion, and also because of the formation of a new type of motion: localized boundary layers with various scales of spatial variation of velocity and density. The thickness of these boundary layers tends to zero as the kinetic coefficients decrease [6, 7].

With allowance for that, the linearized equation (1.3) can be represented in the following operator forms:

$$\hat{L}_4\hat{L}_w f = \hat{L}_2\hat{L}_b f = 0,$$

where $\hat{L}_4$ and $\hat{L}_b$ are singularly disturbed fourth-order operators and $\hat{L}_2$ and $\hat{L}_w$ are regular second-order operators. The general solution of Eq. (1.3) can be represented as a linear combination of solutions of the equations

$$\hat{L}_w f = 0, \qquad \hat{L}_b f = 0, \tag{1.4}$$

where the solutions of the first equation are traveling internal waves and the solutions of the second equation are spatially localized periodic flows, which can be called internal boundary flows by analogy with [7].

The operators introduced above are sought as expansions in powers of the small parameters ν and D. Confining ourselves to the first power, we obtain

$$\hat{L}_4 = \hat{L}_b = \nu D \partial^4 + i(\nu + D)\partial^2 - \omega^2 - \frac{i(\nu + D)}{\omega} k^2 (N^2 + \omega^2),$$

$$\hat{L}_2 = \partial^2 - \frac{2i(\nu + D)}{\omega} k^2 (\mu^2)' \partial + k^2 \Big\{ \mu^2 - \frac{i(\nu + D)}{\omega} [(\mu^2)'' + k^2(1 + \mu^2)^2] \Big\},$$

$$\hat{L}_w = \partial^2 + \frac{2i(\nu + D)}{\omega} k^2 (\mu^2)' \partial + k^2 \Big\{ \mu^2 + \frac{i(\nu + D)}{\omega} [(\mu^2)'' - k^2(1 + \mu^2)^2] \Big\},$$

$$\mu^2(z) = \frac{N^2(z) - \omega^2}{\omega^2}, \qquad \partial \equiv \frac{d}{dz}$$

(the prime denotes derivatives with respect to z). The operators $\hat{L}_4$ and $\hat{L}_b$ coincide in the first order of these expansions, but generally they are not equal. All subsequent results are asymptotic for the small parameters ν and D. Conditions of smallness will be formulated below in Sec. 2.

As follows from (1.3), the function $f(z,k)$ should have continuous derivatives up to the fifth order inclusive for a piecewise-continuous function $N(z)$. This requirement is the boundary condition for propagation of beams of internal waves in the absence of external obstacles and surfaces of discontinuity of the density gradient and its higher derivatives.

If $N(z)$ is a smooth function, the propagation of the beams is described only by the first equation of (1.4), and satisfaction of these boundary conditions is a consequence of the smoothness of $N(z)$ [this is proven by differentiation of the first equation of (1.4)]. If $N(z)$ has discontinuities or discontinuities of derivatives, it is not possible to describe the propagation of the beams only by the first equation of (1.4), and the second equation has to be used. Physically, this means that wave-induced internal boundary flows, whose thickness depends on the kinetic coefficients and wave frequency, arise in the depth of the liquid at the levels of discontinuity of the function $N(z)$ or its derivatives.

Thus, the problem of the propagation of beams of internal waves in an arbitrarily stratified fluid is divided into three main problems: propagation in a medium with a smooth variable stratification $N(z) > \omega$, reflection from the critical level determined by the condition z_c: $N(z_c) = \omega$, and interaction of the beam with discontinuities of $N(z)$ and its derivatives.

2. Propagation of the Beams of Internal Waves. An asymptotic solution of the equation

$$f'' + \frac{2i(\nu + D)}{\omega} k^2 (\mu^2)' f' + k^2 \Big\{ \mu^2 + \frac{i(\nu + D)}{\omega} [(\mu^2)'' - k^2(1 + \mu^2)^2] \Big\} f = 0, \tag{2.1}$$

which describes the propagation of internal waves, can be obtained in the case of a slowly varying function $N(z)$ where the characteristic beam width and the wavelength are much smaller than the scale of variation of the buoyancy frequency $\Lambda_N = |d \ln N(z)/dz|^{-1}$. This is equivalent to the case where the spatial spectrum of the beam $f(z,k)$ is localized near a certain value k_0 that characterizes the wavelength in the beam, and the spectrum width Δk characterizes the beam width. These quantities should satisfy the inequalities $k_0 \Lambda_N \gg 1$ and $\Delta k \Lambda_N \gg 1$.

If additional conditions $k_0^2 \ll \omega/(\nu + D)$ and $(\Delta k)^2 \ll \omega/(\nu + D)$ are satisfied, the asymptotic solution of Eq. (2.1) can be sought in the form $f(z,k) = e^{ik\sigma(z,k)}$. Substituting this expression into (2.1), we obtain the equation for the eikonal $\sigma(z,k)$

$$\sigma'^2 - \frac{i\sigma''}{k} + \frac{2\tilde{\nu} k}{\omega} (\mu^2)' \sigma' - \mu^2 - \frac{i\tilde{\nu}}{\omega} [(\mu^2)'' - k^2(1 + \mu^2)^2] = 0, \tag{2.2}$$

where $\tilde{\nu} = \nu + D$. In turn, in the solving of (2.2), the eikonal is expanded in a power series of the kinetic

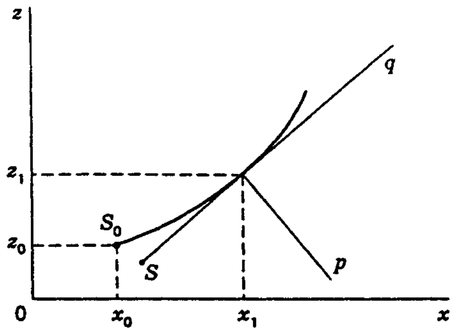

Fig. 1. Geometry of the beam in the laboratory (x, z) and attendant (p, q) systems of coordinates.

coefficients. Retaining terms of the first order of smallness $\sigma' = \sigma_0' + (\tilde{\nu}k^2/\omega)\sigma_1'$ and substituting this into (2.2), we obtain the following system of equations:

$$\sigma_0'^2 - \frac{i\sigma_0''}{k} - \mu^2 = 0, \qquad 2\sigma_0'\sigma_1' - \frac{i\sigma_1''}{k} + \frac{2(\mu^2)'\sigma_0'}{k} + i\left[(1+\mu^2)^2 - \frac{(\mu^2)''}{k^2}\right] = 0. \tag{2.3}$$

Assuming that $\sigma_0' = a_0 + a_1/k + \cdots$, $\sigma_1' = b_0 + b_1/k + \cdots$, we find the coefficients a_i and b_i from (2.3), and, thus, obtain the final expression for $f(z, k)$:

$$f(z,k) = \frac{A(k)}{\sqrt{|\mu|}} \exp\left[\frac{i\tilde{\nu}k^2(1-3\mu^4)}{4\omega\mu^2}\right] \exp\left\{i\lambda k \int\limits_{z_0}^{z} \left[\mu - \frac{i\tilde{\nu}k^2(1+\mu^2)^2}{2\omega\mu}\right] dz'\right\}, \tag{2.4}$$

where the value $\lambda = +1$ corresponds to the beam propagating to the right and down, and $\lambda = -1$ to the beam propagating to the right and up.

It is convenient to represent the field of the beam in an attendant system of coordinates (p, q) where the q axis is tangent to the beam trajectory at the point (x_1, z_1), and the p axis is normal to it (Fig. 1). The relation between the coordinate systems (x, z) and (p, q) is given by the formulas

$$x - x_1 = p\sin\theta + q\cos\theta, \quad z - z_1 = -p\cos\theta + q\sin\theta, \quad \cot\theta = \mu(z_1).$$

Performing this transformation of coordinates in (2.4) and using (1.2), we obtain the following expression for the displacements in the beam:

$$h(p,q) = \sqrt{\frac{\mu(z_0)}{\mu(z_1)}} \int\limits_0^\infty A_0(k) \exp\left\{ikp - \frac{\tilde{\nu}k^3[Q(z_1) + q]}{2N(z_1)\cos\theta}\right\} dk. \tag{2.5}$$

Here $A_0(k)$ is a spectral function determined by the properties of the source of waves located at the point S_0 (see Fig. 1) and $Q(z)$ is the natural *current longitudinal scale of the beam* given by the formula

$$Q(z) = \frac{\mu(z)}{[1+\mu^2(z)]^{3/2}} \int\limits_{z_0}^{z} \frac{[1+\mu^2(z')]^2}{\mu(z')} dz'. \tag{2.6}$$

Its value in an arbitrarily stratified medium differs from the geometric length of the beam

$$L(z) = \int\limits_{z_0}^{z} \sqrt{1+\mu^2(z')}\, dz'$$

and is connected with the latter by the integral relation

$$Q(z) = \frac{\sqrt{L'^2(z) - 1}}{L'^3(z)} \int\limits_{z_0}^{z} \frac{L'^4(z')}{\sqrt{L'^2(z') - 1}} dz',$$

where the prime at $L(z)$ denotes the derivative with respect to the argument. It is easily seen that for $N(z) = \text{const}$ we have $Q(z) = L(z)$.

In an exponentially stratified fluid with a constant buoyancy frequency $N_1 = N(z_1)$, a source located at the point $(p = 0, q = -L_1)$ (the beam is considered in the attendant system of coordinates shown in Fig. 1) radiates a beam whose field, in accordance with [12], has the form

$$h(p,q) = \int_0^\infty A_1(k) \exp\left\{ikp - \frac{\tilde{\nu}k^3(L_1+q)}{2N_1 \cos\theta}\right\} dk. \tag{2.7}$$

A comparison of formulas (2.5) and (2.7) shows that the characteristics of the wave beam radiated at the point S_0 in a medium with a prescribed stratification $N(z)$ in the vicinity of the observation point (x_1, z_1) are the same as for a beam in a medium with a constant buoyancy frequency $N(z_1)$ radiated by a source of the same type located at the point S at a distance $Q(z_1)$ from the observation point [$Q(z)$ is defined by formula (2.6)] with an amplitude increased by a factor of $\sqrt{\mu(z_0)/\mu(z_1)}$.

The study conducted allows one to reduce the problem of the propagation of beams of internal waves in a medium with an arbitrary smoothly varying stratification to the case of a constant buoyancy frequency, which has been studied in detail theoretically [3, 5] and experimentally [12].

3. Reflection on a Critical Level. We consider the propagation of a beam of internal waves near the critical level $z = z_c$, at which the local buoyancy frequency of the medium is compared with the wave frequency $\omega = N(z_c)$. For $z > z_c$ the value of $\mu^2(z)$ becomes negative and, in accordance with (2.4), each spatial harmonic of the beam will decay exponentially. It follows from the conditions of continuity of the function $f(k,z)$ and its derivative for $z = z_c$ that, in addition to the incident beam, two more beams arise: a reflected beam and a beam partly penetrating into a nonwave zone.

It follows from the general reasoning, taking into account (1.2) and (2.4), that for vertical displacements in the incident $h_i(x,z)$, reflected $h_r(x,z)$, and transmitted $h_t(x,z)$ beams we can write the following expressions:

for $z < z_c$,

$$\begin{aligned} h_i(x,z) &= \frac{1}{\sqrt{|\mu|}} \int_0^\infty A(k) \exp\left[\frac{i\tilde{\nu}k^2(1-3\mu^4)}{4\omega\mu^2}\right] \exp\left\{-ik\int_{z_0}^{z}\left[\mu - \frac{i\tilde{\nu}k^2(1+\mu^2)^2}{2\omega\mu}\right]dz\right\} e^{ikx}\, dk, \\ h_r(x,z) &= \frac{1}{\sqrt{|\mu|}} \int_0^\infty B(k) \exp\left[\frac{i\tilde{\nu}k^2(1-3\mu^4)}{4\omega\mu^2}\right] \exp\left\{ik\int_{z_c}^{z}\left[\mu - \frac{i\tilde{\nu}k^2(1+\mu^2)^2}{2\omega\mu}\right]dz\right\} e^{ikx}\, dk; \end{aligned} \tag{3.1}$$

for $z > z_c$,

$$h_t(x,z) = \frac{1}{\sqrt{|\mu|}} \int_0^\infty C(k) \exp\left[\frac{i\tilde{\nu}k^2(1-3\mu^4)}{4\omega\mu^2}\right] \exp\left\{-k\int_{z_c}^{z}\left[|\mu| + \frac{i\tilde{\nu}k^2(1+\mu^2)^2}{2\omega|\mu|}\right]dz\right\} e^{ikx}\, dk.$$

It is required to express the spatial spectra $B(k)$ and $C(k)$ of the reflected and transmitted beams in terms of the spectrum $A(k)$ of the incident beam.

The solutions of (3.1) diverge for $\mu(z_c) = 0$. To match the asymptotics, exact solutions of Eq. (2.1) for $z \sim z_c$ are found and matched with each other.

The function $N(z)$ for $z \sim z_c$ can be represented as a Taylor series expansion with two terms left:

$$N(z) = \omega\left(1 - \frac{z-z_c}{\Lambda_N}\right), \quad \mu^2(z) = -\frac{2(z-z_c)}{\Lambda_N}, \quad \Lambda_N = \left|\frac{1}{N(z)}\frac{dN(z)}{dz}\right|^{-1}_{z=z_c}, \tag{3.2}$$

where Λ_N is the characteristic scale of variation of $N(z)$ near the critical level. Substituting (3.2) into (2.1), we obtain the equation for $f(z,k)$

$$f'' - \frac{4i\tilde{\nu}k^2}{\omega\Lambda_N} f' - k^2\left\{\frac{2(z-z_c)}{\Lambda_N} + \frac{i\tilde{\nu}k^2}{\omega}\left[1 - \frac{4(z-z_c)}{\Lambda_N}\right]\right\} f = 0,$$

which, by means of the substitutions

$$f = \exp\left[\frac{2i\tilde{\nu}k^2(z-z_c)}{\omega\Lambda_N}\right] g(y,k), \qquad y = (z-z_c)\Big(1 - \frac{2i\tilde{\nu}k^2}{\omega}\Big) + \frac{i\tilde{\nu}k^2\Lambda_N}{2\omega},$$

reduces to the Airy equation for the introduced function g (in this equation, the quadratic terms $\sim\tilde{\nu}^2$ are omitted)

$$\frac{d^2g}{dy^2} - \frac{2k^2}{\Lambda_N}\left(1 + \frac{4i\tilde{\nu}k^2}{\omega}\right) yg = 0. \tag{3.3}$$

Taking into account that the incident wave decays as z increases and the reflected wave decays as z decreases, we write solutions of Eq. (3.3) as

$$g_i = \alpha\sqrt{-y}H^{(1)}_{1/3}[\varepsilon(-y)^{3/2}], \quad g_r = \beta\sqrt{-y}H^{(2)}_{1/3}[\varepsilon(-y)^{3/2}], \quad \varepsilon = \frac{2k}{3}\Big(1 + \frac{2i\tilde{\nu}k^2}{\omega}\Big)\sqrt{\frac{2}{\Lambda_N}}, \tag{3.4}$$

where $H^{(1)}_{1/3}$ and $H^{(2)}_{1/3}$ are Hankel functions of the first and second kind. For the transmitted wave, which also decays as z increases, we have

$$g_t = \gamma\sqrt{y}K_{1/3}[\varepsilon y^{3/2}], \tag{3.5}$$

where $K_{1/3}$ is a MacDonald function that differs from the one commonly used [14] by the absence of the factor $\pi/2$.

Ensuring the continuity of the functions $(g_i + g_r)$ and g_t for $z = z_c$ and their first derivatives and retaining terms up to the first degree of $\tilde{\nu}$ inclusive, we obtain

$$\beta = \alpha\,e^{-i\pi/3}, \qquad \gamma = \alpha\,e^{-i\pi/6}. \tag{3.6}$$

Substituting (3.2) into (3.1) and conducting integration, performing an asymptotic expansion for $|\varepsilon y^{3/2}| \gg 1$ in (3.4) and (3.5) and comparing the resultant expressions, we obtain the relationship between the quantities $\{\alpha, \beta, \gamma\}$ and $\{A(k), B(k), C(k)\}$:

$$\alpha = \sqrt{\frac{\pi k}{3}}A(k)\,e^{-i\varphi_0}\,e^{5i\pi/12}, \quad \beta = \sqrt{\frac{\pi k}{3}}B(k)e^{-5i\pi/12}, \quad \gamma = \sqrt{\frac{\pi k}{3}}C(k),$$

where

$$\varphi_0 = k\int_{z_0}^{z_c}\left[\mu - \frac{i\tilde{\nu}k^2(1+\mu^2)^2}{2\omega\mu}\right]dz'.$$

Together with equalities (3.6) this allows us to find

$$B(k) = A(k)\,e^{-i\varphi_0}\,e^{i\pi/2}, \qquad C(k) = A(k)\,e^{-i\varphi_0}\,e^{i\pi/4}. \tag{3.7}$$

It follows from (3.7), in particular, that, being reflected on the critical level, the beam preserves its structure and increases the phase by $\pi/2$.

The beam transmitted into the nonwave zone $z > z_c$ transfers energy that transforms into heat. The technique for calculating the coefficient of energy losses η equal to the ratio of the energy flux passing to the nonwave zone to the energy flux in the incident beam is similar to the calculation of the losses of energy in the case of reflection from a rigid plane, which was described in detail in [6, 7]. Conducting analogous calculations, we obtain

$$\eta = \frac{\tilde{\nu}}{\omega\sin(\pi/3)}\Big(\int_0^\infty k|A(k)|^2e^{-\psi(k)}dk\Big/\int_0^\infty(|A(k)|^2/k)e^{-\psi(k)}dk\Big),$$

where

$$\psi = \frac{\tilde{\nu}k^3}{\omega}\int_{z_0}^{z_c}\frac{(1+\mu^2)^2}{\mu}dz'.$$

Estimation of the coefficient of energy losses for the case of salt stratification under laboratory conditions (where $N \sim 1$ sec^{-1}, $dN/dz \sim 0.1$ sec^{-1}/cm, and $l = 10$ cm) and in the upper atmosphere (the period of buoyancy $T_b \approx 30$ min, $dT_b/dz \approx 5$ min/km, and $l \approx 1$ km) shows that its value is $\eta \approx 10\%$ in the first case and $\eta \approx 0.03\%$ in the second case. Under real conditions, it can be expected in most cases that its value amounts to several percent taking into account the turbulent character of motion, which involves an additional decay of the wave processes.

4. Internal Boundary Flows on Discontinuities of Derivatives of Density. If the buoyancy frequency is a rather smooth function of the z coordinate (it is continuous together with its derivatives up to the fifth order inclusive), the propagation of the wave beams is described by the second-order equation (2.1). When the buoyancy frequency (or its derivatives not higher than the fifth order) is discontinuous, then to ensure the required smoothness of the spectral function, it is necessary to use solutions of the singularly disturbed equation

$$\hat{L}_b f \equiv \nu D f'''' + i\omega(\nu + D) f'' - \Big[\omega^2 + \frac{i(\nu + D)}{\omega} k^2 (N^2 + \omega^2)\Big] f = 0.$$

Its solutions, like those of the wave equation (2.1), can be represented in the eikonal form

$$f_\nu = e^{i\sigma_\nu}, \qquad f_D = e^{i\sigma_D},$$
$$\sigma'_\nu = \pm \frac{1+i}{l_\nu}\Big[1 + \frac{i}{4}\frac{\nu + D}{\nu - D} k^2 l_\nu^2 (\mu^2 + 2)\Big], \quad \sigma'_D = \pm \frac{1+i}{l_D}\Big[1 + \frac{i}{4}\frac{D + \nu}{D - \nu} k^2 l_D^2 (\mu^2 + 2)\Big] \tag{4.1}$$

(the prime denotes the derivative with respect to the z coordinate). Solutions (4.1) include the internal viscous and diffusion scales $l_\nu = \sqrt{2\nu/\omega}$ and $l_D = \sqrt{2D/\omega}$, which characterize the thickness of the split boundary flow. Exactly these scales arise in the problem of reflection of internal waves from a rigid wall [8]. The ratio of these scales is determined by the values of viscosity and diffusivity (the Schmidt number) and does not depend on time, as in the problem of the formation of a diffusion-induced boundary flow on an impermeable wall [15].

Without limiting generality, we can assume that the discontinuity of the buoyancy frequency (or its derivatives) lies at the level $z = 0$. Let the buoyancy frequency be a smooth function $N_1(z)$ below this level and a smooth function $N_2(z)$ above this level, and $\mu_i(z) = \sqrt{N_i^2(z) - \omega^2}/\omega$.

In the vicinity of the discontinuity of the buoyancy frequency (or its higher derivative), the flow pattern that is formed when the wave beam is incident onto this discontinuity from below includes the incident beam itself, a beam transmitted upward, the beam transmitted downward, and two pairs of boundary flows (below and above the discontinuity) with split scales of velocity and density (salinity) variation. The spatial structure of the wave beams is described by formulas (1.2) and (2.4), and that of the boundary flows by formulas (4.1). Thus, the spectral functions of the complete field can be written as

$$f = Ae^{i\sigma_1^+} + Be^{i\sigma_1^-} + D_{1\nu}e^{i\sigma_{1\nu}} + D_{1D}e^{i\sigma_{1D}}, \ z < 0, \quad f = Ce^{i\sigma_2} + D_{2\nu}e^{i\sigma_{2\nu}} + D_{2D}e^{i\sigma_{2D}}, \ z > 0.$$

Here A, B, and C are the amplitudes of the incident, transmitted, and reflected waves, $D_{1\nu}$ and $D_{2\nu}$ are the amplitudes of the velocity boundary layers above and below the boundary of the discontinuity, and D_{1D} and D_{2D} are the amplitudes of the density boundary layers. In accordance with the previous results, the eikonals are expressed as

$$\sigma_1^{+\prime} = \varphi_{11}^+ + k^2(l_\nu^2 + l_D^2)\varphi_{12}^+, \qquad \sigma_1^{-\prime} = \varphi_{11}^- + k^2(l_\nu^2 + l_D^2)\varphi_{12}^-, \quad \sigma_2' = \varphi_{21} + k^2(l_\nu^2 + l_D^2)\varphi_{22},$$

$$\sigma'_{1\nu} = -\frac{1+i}{l_\nu}(1 + ik^2 l_\nu^2 \psi_1), \quad \sigma'_{2\nu} = \frac{1+i}{l_\nu}(1 + ik^2 l_\nu^2 \psi_2),$$

$$\sigma'_{1D} = -\frac{1+i}{l_D}(1 - ik^2 l_D^2 \psi_1), \quad \sigma'_{2D} = \frac{1+i}{l_D}(1 - ik^2 l_D^2 \psi_2).$$

The prime denotes the derivative with respect to the z coordinate, and the following designations are introduced:

$$\varphi_{11}^\pm = \mp k\mu_1 + \frac{i\mu_1'}{2\mu_1}, \quad \varphi_{12}^\pm = \pm\frac{ik(1 + \mu_1^2)^2}{4\mu_1} - \frac{(1 + 3\mu_1^4)\mu_1'}{4\mu_1^3}, \quad \varphi_{21} = -k\mu_2 + \frac{i\mu_2'}{2\mu_2},$$

$$\varphi_{22} = \frac{ik(1+\mu_2^2)^2}{4\mu_2} - \frac{(1+3\mu_2^4)\mu_2'}{4\mu_2^3}, \quad \psi_1 = \frac{1}{4}\frac{\nu+D}{\nu-D}(2+\mu_1^2), \quad \psi_2 = \frac{1}{4}\frac{\nu+D}{\nu-D}(2+\mu_2^2).$$

In these equations, the functions $\mu_i(z)$ and their derivatives are taken at the point $z = 0$.

The continuity of the function f and its derivatives up to the fifth order inclusive leads to the following sixth-order algebraic system for determining the unknown amplitudes:

$$(e^{i\sigma_1^-})^{(n)}B - (e^{i\sigma_2})^{(n)}C + (e^{i\sigma_{1\nu}})^{(n)}D_{1\nu} + (e^{i\sigma_{1D}})^{(n)}D_{1D} - (e^{i\sigma_{2\nu}})^{(n)}D_{2\nu} - (e^{i\sigma_{2D}})^{(n)}D_{2D}$$
$$= -(e^{i\sigma_1^+})^{(n)}A_0, \qquad n = 0, 1, \ldots, 5, \tag{4.2}$$

Here (n) is the corresponding derivative with respect to z for $z = 0$ and $A_0 = A\exp\left(i\int_{z_0}^{0}\sigma_1^{+\prime}\,dz\right)$. The elements of the matrix of system (4.2) and the elements of the vectors of the right-hand sides were written as finite expansions with respect to the degrees of the small parameters l_ν and l_D. A solution of the system of equations (4.2) was sought as an expansion of the amplitudes B, C, $D_{1\nu}$, $D_{2\nu}$, D_{1D}, and D_{2D} in series in powers of l_ν and l_D. For simplicity, below we present the solutions of this system in the absence of salt diffusion:

$$B = \frac{k(\mu_1-\mu_2) - (i/2)(\mu_1'/\mu_1 - \mu_2'/\mu_2)}{\Delta} - \frac{ic}{2\Delta^2}, \quad C = \frac{2k\mu_1}{\Delta} - \frac{id}{2\Delta^2},$$
$$D_{1\nu} = \frac{ia}{4\Delta} + \frac{(1+i)b}{8\Delta}, \qquad D_{2\nu} = -\frac{ia}{4\Delta} + \frac{(1+i)b}{8\Delta},$$

where

$$\Delta = k(\mu_1+\mu_2) + \frac{i}{2}\left(\frac{\mu_1'}{\mu_1} - \frac{\mu_2'}{\mu_2}\right), \quad a = k\mu_1\left[2k^2(\mu_1^2-\mu_2^2) - \frac{3}{2}\left(\frac{\mu_1'^2}{\mu_1^2} - \frac{\mu_2'^2}{\mu_1^2}\right) + \frac{\mu_1''}{\mu_1} - \frac{\mu_2''}{\mu_2}\right],$$
$$b = \mu_1\mu_2k^2\left[2k^2(\mu_1^2-\mu_2^2) - \frac{3}{2}\left(\frac{\mu_1'^2}{\mu_1^2} - \frac{\mu_2'^2}{\mu_2^2}\right) + \left(\frac{\mu_1''}{\mu_1} - \frac{\mu_2''}{\mu_2}\right)\right] - ik\mu_1[s + 3k^2(\mu_1\mu_1' - \mu_2\mu_2')],$$
$$c = \frac{k^4(\mu_1^2-\mu_2^2)(\mu_1^2\mu_2^2-1)}{\mu_1\mu_2} + ik^3\mu_1\left[\frac{\mu_1'}{\mu_1^3} - \frac{\mu_2'}{\mu_2^3} - \frac{\mu_2'(\mu_1^2-\mu_2^2)}{\mu_2}\right] - ik\mu_1\left[s - \frac{3\mu_2'}{4\mu_2}\left(\frac{\mu_1'^2}{\mu_1^2} - \frac{\mu_2'^2}{\mu_2^2}\right) + \frac{\mu_2'}{2\mu_2}\left(\frac{\mu_1''}{\mu_1} - \frac{\mu_2''}{\mu_2}\right)\right],$$
$$d = -k^4(\mu_1^2-\mu_2^2)\left[\frac{\mu_1^2\mu_2^2+1}{\mu_1\mu_2} + 2\mu_1^2\right] + ik^3\mu_1\left[\frac{\mu_1'}{\mu_1^3} - \frac{\mu_2'}{\mu_2^3} - \frac{\mu_1'(\mu_1^2-\mu_2^2)}{\mu_1}\right]$$
$$+ k^2\mu_1(\mu_1+\mu_2)\left[\frac{3}{2}\left(\frac{\mu_1'^2}{\mu_1^2} - \frac{\mu_2'^2}{\mu_2^2}\right) - \frac{\mu_1''}{\mu_1} + \frac{\mu_2''}{\mu_2}\right] - ik\mu_1\left[s - \frac{3\mu_1'}{4\mu_1}\left(\frac{\mu_1'^2}{\mu_1^2} - \frac{\mu_2'^2}{\mu_2^2}\right) + \frac{\mu_1'}{2\mu_1}\left(\frac{\mu_1''}{\mu_1} - \frac{\mu_2''}{\mu_2}\right)\right],$$
$$s = \frac{15}{4}\left(\frac{\mu_1'^3}{\mu_1^3} - \frac{\mu_2'^3}{\mu_2^3}\right) - \frac{9}{2}\left(\frac{\mu_1'\mu_1''}{\mu_1^2} - \frac{\mu_2'\mu_2''}{\mu_2^2}\right) + \frac{\mu_1'''}{\mu_1} - \frac{\mu_2'''}{\mu_2}.$$

It follows from these formulas that the character of the flow that arises significantly depends on the fine structure of the density field of the medium. If the buoyancy frequency and/or its first derivative are discontinuous, the amplitude of the reflected beam has the same order as the amplitude of the incident beam, whereas the amplitude of the boundary flow is small and is determined by the viscosity of the medium ($\sim\nu$). If the buoyancy frequency and its first derivative are continuous but its second derivative is discontinuous, then both the reflected beam and the boundary layer have the same order of smallness $\sim\nu$. If only the third derivative of the buoyancy frequency is discontinuous, the reflected beam has order of smallness $\sim\nu$, and for the boundary flow it is $\sim\nu^{3/2}$. Finally, if the buoyancy frequency is continuous together with its derivative up to the third, there are neither reflected beam nor internal boundary flow in accordance with the results of Secs. 1 and 2.

Thus, the direct calculations show that, on discontinuities of the density gradient, the transmitted beam of internal waves does not simply scatter but forms specific internal boundary flow with split scales of velocity and density variation. The ratio of the fraction of energy that remains in the wave field to the one that passes to the boundary flow changes, depending on the character of discontinuity. The degree of localization of the boundary flow increases with increase in the wave frequency.

In the linear approximation, the incident wave and the flows induced by it do not interact with one another. Under real conditions, because of the nonlinearity of the equations of motion and the equation of state, this interaction occurs and generates new forms of motion with their own characteristic scales. Qualitatively, we can identify at least two classes of these scales: a macroscale, determined by the geometry of the wave beam and a microscale, determined by the minimum kinetic coefficient of the problem (in this case, by the diffusivity) and the wave frequency. The stage of destruction of nonlinearly interacting internal waves is preceded by the formation of small-scale high-gradient interlayers (discontinuities of stratification [16]), which is indicative of the importance of the effects of scale splitting in the general wave dynamics.

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